TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 259, Number 1, May 1980

ZEROS OF SUCCESSIVE DERIVATIVES OF ENTIRE FUNCTIONS OF THE FORM $h(z)\exp(-e^z)$

BY

ALBERT EDREI[1]

ABSTRACT. Consider $f(z) = h(z)\exp(-e^z)$ $(z = x + iy)$ where $h(z)$ is a real entire function of finite order having no zeros in some strip $\{x + iy: |y-\pi| < \eta_1, x > x_0\}$ $(0 < \eta_1)$. The author studies the power series (1) $f(\tau + z) = \sum_{n=0}^{\infty} a_n z^n$ (τ real) and the number $N(\tau_1, \tau_2; n)$ of real zeros of $f^{(n)}(z)$ which lie in the interval $[\tau_1, \tau_2]$. He proves (2) $N(\tau_1, \tau_2; n) \sim (\tau_2-\tau_1)n/(\log n)^2$ $(n \to \infty)$. With regard to the expansion (1) he determines a positive, strictly increasing, unbounded sequence $\{\nu_k\}_{k=1}^{\infty}$ such that $(\nu_{k+1}-\nu_k)/\log \nu_k \to 1$ $(k \to \infty)$ and having the following properties: (i) if $\nu_k < n < \nu_{k+1}$, then $a_n \neq 0$ and all the a_n have the same sign; (ii) if in addition $\nu_{k+1} < m < \nu_{k+2}$, then $a_m a_n < 0$. It is possible to deduce from (2) the complete characterization of the final set (in the sense of Pólya) of $\exp(-e^z)$.

Introduction. We consider functions of the form

$$f(z) = h(z)\exp(-e^z) \qquad (z = x + iy), \tag{1}$$

where $h(z)$ is a *real entire function of finite order* satisfying the following additional condition:

I. *There are no zeros of* $h(z)$ *in some strip*

$$\{x + iy: |y-\pi| < \eta_1, x > x_0\}. \tag{2}$$

The fixed quantity η_1 *is positive, otherwise arbitrary.*

There are no further restrictions on the zeros of $h(z)$.

Since we may select $\eta_1 < \pi$, it is clear that functions such as $\sin z$ or $\sin(z^2)$ having infinitely many positive zeros, represent admissible choices of $h(z)$.

This paper is devoted to the study of the power series expansion

$$f(\tau + z) = \sum_{n=0}^{\infty} a_n z^n \qquad (\tau \text{ real}), \tag{3}$$

and to the evaluation of the number of real zeros of $f^{(n)}(z)$ in some given interval $[\tau_1, \tau_2]$.

Let $F(z)$ be an entire, transcendental function. Since the pioneering work of Laguerre and Borel, the study of the distribution of zeros of the successive derivatives

$$F'(z), F''(z), \ldots, F^{(n)}(z), \ldots \tag{4}$$

has been the center of many important investigations.

Presented to the Society April 20, 1979; received by the editors March 13, 1979.

AMS (MOS) subject classifications (1970). Primary 30A66; Secondary 30A70.

[1]The research of the author was supported by a grant from the National Science Foundation Grant MCS-7701278.

Among the recent remarkable contributions to the subject, explicit mention should be made of the proof by Hellerstein and Williamson [2], [3] of a conjecture of Pólya and Wiman unresolved since 1915.

DEFINITION OF THE FINAL SET $S(F)$. *Following Pólya we define the final set $S(F)$ of $F(z)$ to be the set of all limit points of all the zeros of all the functions* (4).

It is understood that $z_0 \in S(F)$ if $F^{(n)}(z_0) = 0$ for infinitely many values of n.

In an address presented before the American Mathematical Society (1942), Pólya [5, p. 182] asserted that, in an unpublished paper, Szegö had shown that the final set of

$$g(z) = \exp(-e^{e^z}) \tag{5}$$

consists of an infinity of parallel lines, dividing the plane into congruent strips of width 2π.

In a recent letter, Pólya asked the present author whether he could supply a proof of this fact. It appears that Szegö cannot find, in his notes, any trace of such a proof and considers it to be irremediably lost. Neither Pólya nor Szegö feel inclined to undertake a new study of the question.

I believe that the form (5) which appears in the printed version of Pólya's survey [5, p. 182] contains a misprint and that Pólya and Szegö meant

$$g_1(z) = \exp(-e^z). \tag{6}$$

The latter function is of the form considered in (1) and is consequently covered by the following.

THEOREM 1. *Let $f(z)$ be any entire function of the form* (1) *and let τ_1 and τ_2 be real quantities such that $-\infty < \tau_1 < \tau_2 < +\infty$, otherwise arbitrary.*

Denote by $N(\tau_1, \tau_2; n)$ the number of real zeros of $f^{(n)}(z)$ which lie in the fixed interval $[\tau_1, \tau_2]$. Then, if n is large enough, all these zeros are simple and satisfy the relation

$$\left| N(\tau_1, \tau_2; n) - \frac{n(\tau_2-\tau_1)}{(\log n - \log\log n)^2} \right| < \frac{B(\tau_2-\tau_1)n}{(\log n)^3}, \tag{7}$$

where $B = A \max(1, |\tau_1|, |\tau_2|)$ ($0 < A$ = absolute constant).

It is clear that (7) implies the less precise but strikingly simple asymptotic relation

$$N(\tau_1, \tau_2; n) \sim \frac{n(\tau_2-\tau_1)}{(\log n)^2} \qquad (n \to \infty). \tag{8}$$

The influence of the factor $h(z)$ cannot be perceived in the relations (7) and (8). A more thorough study of a certain logarithmic derivative $(1/J)(\partial J/\partial\tau)$ (it appears in our formula (7.21)) would enable us to isolate and assess this influence. For the sake of brevity this aspect of the question will be omitted.

Since Theorem 1 is applicable to the function $g_1(z)$ in (6), every point of the real axis belongs to the final set $S(g_1)$. Moreover, in view of the periodicity $g_1(z) \equiv$

$g_1(z + 2\pi i)$, all the lines

$$y = 2k\pi i \qquad (z = x + iy;\ k = \pm 1, \pm 2, \pm 3, \dots)$$

also belong to $S(g_1)$. The proof that $S(g_1)$ has no other points is a simple consequence of the well-known elementary fact [6, p. 47, ex. 62] that

$$g_1^{(n)}(z) = P_n(w)e^{-w} \qquad (w = e^z),$$

where $P_n(w)$ is a polynomial in w whose zeros are all real and nonnegative.

We thus see that the final set $S(g_1)$ exactly fits the description given by Pólya of $S(g)$ and therefore strengthens the conjecture that the particular form of $g(z)$ (in (5)) is due to a misprint.

The problem of finding the final set of a given entire function $F(z)$ is one of considerable complexity. As far as I know, the preceding construction of $S(g_1)$ represents the first actual determination of the final set of a *given* entire function of infinite order.

Surprisingly enough the following converse problem admits of a very simple solution.

CONVERSE PROBLEM. *Let S be a given set such that $\infty \in S$ and such that the image of S on the complex sphere is compact. Determine an entire function $F(z)$ such that $S(F) = S$.*

Edrei and MacLane [1] have shown that it is always possible to find a solution of the converse problem having some preassigned order λ $(0 < \lambda < +\infty)$.

Quite recently, Prather [7] has studied the power series expansion of functions closely related to $g_1(z)$. He states one of his results as follows: in the expansion $\exp(-e^z) = \sum_{n=0}^{\infty} a_n z^n$, more than 97 percent of the coefficients a_n are not zero. Prather conjectures that, in fact, $a_n \neq 0$ for n large enough.

Suitably modified, the method which leads to Theorem 1 also yields some relevant information concerning Prather's problem. I prove:

THEOREM 2. *Let $f(z)$ be of the form* (1) *and let* (3) *be its Taylor expansion where τ is real, otherwise arbitrary.*

It is then possible to find a sequence

$$\nu_1, \nu_2, \nu_3, \dots \tag{9}$$

positive, strictly increasing, unbounded and having all the following properties.

I. *As $k \to \infty$,*

$$k \sim \nu_k / \log \nu_k, \tag{10}$$

and more precisely,

$$(\nu_{k+1} - \nu_k)/\log \nu_k \to 1. \tag{11}$$

II. *If $n > \nu_1$, then $a_n = 0$ if and only if*

$$n = \nu_k \tag{12}$$

and ν_k is an integer.

III. *For all n such that*

$$\nu_k < n < \nu_{k+1}, \tag{13}$$

the coefficients a_n are not zero, and they all have the same sign.

IV. *If*

$$\nu_{k+1} < m < \nu_{k+2}, \tag{14}$$

then

$$a_n a_m < 0. \tag{15}$$

I prove the existence of the sequence (9) by a continuity argument which gives me no information on the possible appearance of integers among its members. Hence my method does not settle Prather's conjecture. On the other hand, the expansions which may be treated are somewhat more general than those considered by Prather and, for all of them, (10) and (11) show that, in a strong and precise sense, 100 percent of the coefficients are not zero.

As might be expected, my method also gives good approximations for $|a_n|$. These approximations are contained in formulae (8.1) and (8.2) [§8 of the present paper].

The asymptotic behavior of the coefficients G_n (Bell numbers) in the expansion

$$\exp(e^z - 1) = \sum_{n=0}^{\infty} \frac{G_n}{n!} z^n, \tag{16}$$

has been obtained, with complete success, by Moser and Wyman [4].

In view of the close connection between the left-hand side of (16) and $f(z)$ in (1), it is not surprising that the principle of my proofs does not differ from the simple and elegant one adopted by Moser and Wyman. The present paper is not a completely trivial extension of [4] because I need to locate the zeros of $f^{(n)}(z)$. This makes it necessary to study some asymptotic expressions as z varies (Theorem 1) and n varies (Theorem 2; to obtain the latter result n is to be considered as a continuous variable).

The possibility of using my approach to treat other questions about zeros of successive derivatives will be illustrated by the following.

THEOREM 3. *Let $F_1(z)$ be a transcendental entire function of the form*

$$F_1(z) = P(z) e^{\gamma z} \prod_j \left(1 + \frac{z}{\alpha_j}\right)$$

where $\gamma \geq 0$, $\alpha_j > 0$, $\sum_j \alpha_j^{-1} < +\infty$, and $P(z)$ is a polynomial, real for real values of z and positive for large positive values of z. Consider $f_1(z) = \exp(-F_1(z))$ and let $N(\tau_1, \tau_2; n)$ be the number of zeros of $f_1^{(n)}(z)$ in the interval $[\tau_1, \tau_2]$.

If τ_1 and τ_2 are fixed and if the positive sequence $\{x_n\}_n$ is defined by the condition $F_1(x_n) = n$ $(n > n_0)$, then

$$N(\tau_1, \tau_2; n) \sim \frac{(\tau_2 - \tau_1) n}{x_n^2 \{F_1'(x_n)/F_1(x_n)\}} \qquad (n \to \infty).$$

Suitably extended my proof of Theorem 1 will also show that all functions of the form (1) have exactly the same final set as $\exp(-e^z)$.

A detailed proof of Theorem 3 and the construction of the final set of $h(z)\exp(-e^z)$ would inordinately lengthen this paper; they are consequently omitted. Since other more general theorems seem accessible by my method, I expect to return to the question on some future occasion.

1. Outline of the method and notational conventions. Throughout the paper τ is a real parameter subject to the restriction

$$|\tau| \leq D \qquad (D < +\infty), \tag{1.1}$$

otherwise arbitrary.

We set

$$H(\varphi) = -e^{\tau} \exp(\rho e^{i\varphi}) - in\varphi \tag{1.2}$$

where $\rho > 0$ is a parameter and φ a real variable. It is unnecessary to specify, in the notation, the dependence of H on the real variables τ, ρ and n.

We define

$$t = \rho e^{i\varphi} \tag{1.3}$$

and consider the partial derivatives

$$\partial^j H / \partial \varphi^j = H^{(j)}(\varphi) \qquad (j = 1, 2, 3).$$

We note, for later use,

$$H'(\varphi) = -i(e^{\tau} t e^{t} + n), \tag{1.4}$$

$$H''(\varphi) = e^{\tau} t e^{t}(1 + t), \tag{1.5}$$

$$H'''(\varphi) = i e^{\tau} t e^{t}(1 + 3t + t^2). \tag{1.6}$$

Cauchy's formula for the derivatives yields

$$f^{(n)}(\tau) = \frac{n!}{\pi \rho^n} \operatorname{Re} \int_0^{\pi} h(\tau + \rho e^{i\varphi}) e^{H(\varphi)}\, d\varphi. \tag{1.7}$$

Our method requires:

(i) the adequate selection of

$$\rho = \rho(\tau, n) > 0; \tag{1.8}$$

(ii) the choice of a point

$$\zeta = \zeta(\tau, n) = \rho(\tau, n) e^{i\Psi(\tau, n)} \qquad (0 < \Psi < \pi), \tag{1.9}$$

such that

$$\zeta e^{\zeta} = -e^{-\tau} n \qquad (\zeta = \zeta(\tau, n)); \tag{1.10}$$

(iii) a straightforward asymptotic computation based on the fact that (1.4) and (1.10) imply

$$H'(\Psi) = 0, \tag{1.11}$$

and consequently

$$H(\varphi) - H(\Psi) = (H''(\Psi)/2!)(\varphi - \Psi)^2 + \cdots \tag{1.12}$$

with

$$H''(\Psi) = -n(1 + \zeta). \tag{1.13}$$

In view of the relations (1.10)–(1.13), the point ζ defined by (1.9) is a "saddle point".

The well-known properties of this point enable us to follow the familiar pattern known as "method of steepest descent". (A lucid description of the method is given by Szegö [**8**, p. 221].) The various steps prescribed by this approach are taken in

§§2–7. Rewrite (1.7) in the form

$$f^{(n)}(\tau) = (n!/\pi\rho^n)\mathrm{Re}\{e^{H(\Psi)}J(\tau, n)\} \tag{1.14}$$

with

$$J(\tau, n) = \int_0^\pi h(\tau + \rho e^{i\varphi})\exp(H(\varphi)-H(\Psi))\, d\varphi. \tag{1.15}$$

The formula (1.14) is valid only for positive integral values of n. On the other hand, (1.15) involves an integral whose meaning and behavior are unaffected by the fact that n is an integer. Until the very last stages of the proofs (§§8, 9, 10, 11), we shall be concerned only with the study of $J(\tau, n)$ and, consequently, may assume that n is a positive continuous variable. Partial derivatives with respect to n will frequently occur.

It is convenient to introduce ξ as a continuous positive variable and to set

$$\xi = e^{-\tau}n, \tag{1.16}$$

where τ and n (> 0) are continuous real variables. Clearly

$$\partial\xi/\partial n = e^{-\tau}, \qquad \partial\xi/\partial\tau = -\xi. \tag{1.17}$$

Our asymptotic formulae will always refer to $n \to +\infty$. By (1.1) and (1.16), $\xi \to +\infty$ and $n \to +\infty$ are equivalent statements.

The error terms will, according to the context, be denoted by

$$\varepsilon(n), \quad \varepsilon(\xi), \quad \varepsilon(\zeta), \tag{1.18}$$

and we usually find it unnecessary to remind the reader of the fact that $n \to +\infty$.

The symbols (1.18) are to be understood as follows: given $\eta > 0$, there exists some $n_0 = n_0(\eta)$ such that $n > n_0$ implies

$$|\varepsilon| < \eta. \tag{1.19}$$

For the sake of clarity *we shall only use the above notation if inequality* (1.19) *holds uniformly for all* τ *restricted by* (1.1). The symbols ($n > n_0$) following some relation mean that the relation in question is valid for all $n > n_0$ and all τ restricted by (1.1).

Throughout the paper the quantity ε, which may be complex, does not have the same value at each occurrence. The symbol A denotes a positive absolute constant; $K > 0$ is a constant which may depend on several parameters and ω always denotes a complex quantity such that $|\omega| < 1$. The quantities n_0, A, K, ω may have different values in different places.

2. The saddle point. In view of certain applications that the author intends to present elsewhere, it is convenient to solve the equation in (1.10) in slightly more general form. Consider the equation

$$\zeta e^{\zeta} = -u \tag{2.1}$$

where u is a complex variable:

$$u = \xi e^{-i\mu} \qquad (\xi > 0, \mu \text{ real}). \tag{2.2}$$

Lemma 2.1. *Let* $\xi > e$ *and*

$$s = (\log\log\xi - i(\pi-\mu))/(1 + \log\xi). \tag{2.3}$$

If

$$|s| < 2-\sqrt{3} \tag{2.4}$$

equation (2.1) *has a solution of the form*

$$\zeta = \log \xi - \log \log \xi + i(\pi-\mu) + \Phi, \tag{2.5}$$

where

$$\Phi = \sum_{j=1}^{\infty} c_j s^j \qquad (c_1 = 1), \log \xi > 1, \log \log \xi > 0. \tag{2.6}$$

The coefficients $c_j = c_j(\xi)$ *are polynomials in the variable* $(1 + \log \xi)^{-1}$ *such that*

$$|c_j(\xi)| < (2-\sqrt{3})^{-j} \qquad (j = 1, 2, 3, \ldots). \tag{2.7}$$

PROOF. From (2.1), (2.2), (2.3) and (2.5) we deduce, after some elementary reductions,

$$\Phi = s + \frac{\log \xi}{1 + \log \xi} \sum_{j=2}^{\infty} \frac{(-1)^j \Phi^j}{j!}. \tag{2.8}$$

Cauchy's comparison method known as "calcul des limites" suggests the introduction of the dominant relation

$$\Phi_1 = s_1 + \frac{\Phi_1^2}{2(1-\Phi_1)},$$

which is solved by the power series

$$\Phi_1 = \sum_{j=1}^{\infty} C_j s_1^j = \frac{1}{3}\left\{1 + s_1 - (1-s_1)\left(1 - \frac{2s_1}{(1-s_1)^2}\right)^{1/2}\right\},$$

whose coefficients C_j are all positive.

The results of Cauchy imply the existence of a power series such as (2.6) satisfying identically in s the relation (2.8). There are no difficulties about convergence because Cauchy's method yields $|c_j| < C_j$ $(j > 1)$, and, if (2.4) is satisfied,

$$\sum_{j=1}^{\infty} |c_j|\, |s|^j < \sum_{j=1}^{\infty} C_j (2-\sqrt{3})^j < 1.$$

The polynomial character of the coefficients $c_j(\xi)$ is also contained in the assertions of Cauchy.

From this point on, $\zeta(u)$ means the unique solution of (2.1) given by

$$\zeta(u) = \zeta(\xi e^{-i\mu}) = \log \xi - \log \log \xi + i(\pi-\mu) + \sum_{j=1}^{\infty} c_j s^j, \tag{2.9}$$

with μ fixed, $\xi > \xi_0 > e$ and s defined by (2.3). The fact that there are infinitely many solutions of (2.1) need not concern us; we start from the value $\zeta(\xi e^{-i\mu})$ deduced from (2.9) for some large value of $\xi = \xi_0$ and follow it by continuity as ξ increases to $+\infty$. As ξ varies (and μ remains fixed) (2.9), (2.7), (2.3) and the polynomial character of $c_j(\xi)$ show that $\zeta(\xi e^{-i\mu})$ describes an analytic curve $\mathcal{C}$ which has the line $\mathrm{Im}(z) = \pi-\mu$ for asymptote.

We now set

$$\zeta(u) = x(u) + iy(u) = \rho(u)e^{i\Psi(u)} \qquad (x, y \text{ real}, \rho > 0) \tag{2.10}$$

and select for $\Psi(u)$ the determination

$$|\Psi(u)| < \pi/2 \qquad (\xi > \xi_0). \tag{2.11}$$

Differentiations with respect to the positive variable ξ will be denoted by a dot. Hence from (2.1)

$$\frac{d\zeta}{du} = \frac{1}{u}\left(1 - \frac{1}{1+\zeta}\right), \tag{2.12}$$

$$\dot{\zeta} = \frac{d\zeta}{du}\frac{\partial u}{\partial \xi} = \frac{1}{\xi}\left(1 - \frac{1}{1+\zeta}\right). \tag{2.13}$$

All the following relations are now obvious and will be stated without proof:

$$x(u) = \rho(u)\cos \Psi(u) = \log \xi - \log\log \xi + \frac{\log\log\xi}{1+\log\xi} + \omega A\left(\frac{\log\log\xi + |\pi-\mu|}{1+\log\xi}\right)^2, \qquad (\xi > \xi_0), \tag{2.14}$$

$$y(u) = \rho(u)\sin \Psi(u) = \pi - \mu - \frac{\pi-\mu}{1+\log\xi} + \omega A\left(\frac{\log\log\xi + |\pi-\mu|}{1+\log\xi}\right)^2 \qquad (\xi > \xi_0), \tag{2.15}$$

$$\pi - \mu = \Psi(u) + y(u), \tag{2.16}$$

$$\rho(u)e^{x(u)} = \xi, \tag{2.17}$$

$$\rho(u) \sim \log \xi \qquad (\xi \to \infty). \tag{2.18}$$

From (2.13)

$$\dot{x} = \frac{1}{\xi}\left(1 - \frac{1 + x(u)}{|1+\zeta(u)|^2}\right), \tag{2.19}$$

and

$$\dot{y} = \frac{y(u)}{\xi}\frac{1}{|1+\zeta(u)|^2} = -\dot{\Psi}. \tag{2.20}$$

From (2.17) and (2.19)

$$\frac{\dot{\rho}}{\rho} = \frac{1}{\xi}\frac{1+x(u)}{|1+\zeta(u)|^2}, \tag{2.21}$$

$$\dot{\rho} \sim 1/\xi \qquad (\xi \to \infty). \tag{2.22}$$

Throughout the remainder of the paper we always set $\mu = 0$ so that $u = \xi$.

3. The integral $J(\tau, n)$ and its partial derivatives. Consider (1.15) and note that, by definition

$$H(\varphi) - H(\Psi) = -e^{\tau}\{\exp(\rho(\xi)e^{i\varphi}) - \exp(\zeta(\xi))\} - in(\varphi - \Psi(\xi)). \tag{3.1}$$

In order to evaluate $\partial J/\partial n$ and $\partial J/\partial \tau$ we introduce

$$w(\xi) = \exp(t(\xi)) - \exp(\zeta(\xi)) \qquad (t(\xi) = \rho(\xi)e^{i\varphi}),$$

and deduce from (2.1) and (2.10) (with $u = \xi$)

$$\dot{w}(\xi) = (\dot{\rho}/\rho)(te^t - \zeta e^\zeta) + i\xi\dot{\Psi}. \tag{3.2}$$

Set

$$\frac{\partial J(\tau, n)}{\partial n} = \int_0^\pi W(\tau, n, \varphi)\exp(H(\varphi) - H(\Psi))\, d\varphi, \tag{3.3}$$

$$\frac{\partial J(\tau, n)}{\partial \tau} = \int_0^\pi Z(\tau, n, \varphi)\exp(H(\varphi) - H(\Psi))\, d\varphi. \tag{3.4}$$

Taking (3.2) into account we find

$$W(\tau, n, \varphi) = -h(\tau + t)\left[\frac{\dot{\rho}}{\rho}(te^t - \zeta e^\zeta) + i(\varphi - \Psi)\right] + e^{-\tau}\dot{\rho}e^{i\varphi}h'(\tau + t), \tag{3.5}$$

and

$$Z(\tau, n, \varphi) = h(\tau + t)\left[n(\dot{\rho}/\rho)(te^t - \zeta e^\zeta) - e^\tau(e^t - e^\zeta)\right] + (1 - \dot{\rho}\xi e^{i\varphi})h'(\tau + t). \tag{3.6}$$

4. Variation of $H(\varphi)$. Put

$$H(\varphi) = G(\varphi) + iL(\varphi) \qquad (G, L, \text{ real}). \tag{4.1}$$

By (1.2) we find

$$G(\varphi) = -e^\tau \exp(\rho \cos \varphi)\cos(\rho \sin \varphi) \tag{4.2}$$

and, in view of (1.10),

$$G(\varphi) = -(n/\rho)\exp(\rho(\cos \varphi - \cos \Psi))\cos(\rho \sin \varphi), \tag{4.3}$$

$$H(\Psi) = (n/\zeta) - in\Psi. \tag{4.4}$$

Hence

$$G(\Psi) = (n/\rho)\cos \Psi, \tag{4.5}$$

$$L(\Psi) = n\{\operatorname{Im}(1/\zeta) - \Psi\} = n\{\operatorname{Im}(\zeta + 1/\zeta) - \pi\}. \tag{4.6}$$

[To obtain the latter expression we have used (2.16).]

From (1.4) and (1.10) we deduce

$$G'(\varphi) = -n \exp\{\rho(\cos \varphi - \cos \Psi)\}\sin((\varphi - \Psi) + \rho(\sin \varphi - \sin \Psi)),$$

and hence, taking (2.16) into account,

$$G'(\varphi) > 0 \qquad (0 < \varphi < \Psi),$$
$$G'(\varphi) < 0 \qquad (\Psi < \varphi < 2\Psi). \tag{4.7}$$

We now prove

$$\int_{2\Psi}^{\pi} \exp(G(\varphi) - G(\Psi))\, d\varphi < \pi \exp\left(-\frac{3n}{(\log n)^2}\right) \qquad (n > n_0). \tag{4.8}$$

An inspection of (4.3) shows that

$$G(\varphi) < |G(\varphi)| < (n/\rho)\exp(\rho(\cos \varphi - \cos \Psi)) < (n/\rho)\exp(\rho(\cos 2\Psi - \cos \Psi)) \qquad (2\Psi < \varphi < \pi),$$

and using (4.5) we find

$$G(\varphi) - G(\Psi) < (n/\rho)\{\exp(\rho(\cos 2\Psi - \cos \Psi)) - \cos \Psi\} \qquad (2\Psi < \varphi < \pi). \tag{4.9}$$

By elementary considerations and (2.16)

$$\exp(\rho(\cos 2\Psi-\cos \Psi))-\cos \Psi < \exp(-\rho\Psi \sin \Psi)-\cos \Psi$$

$$< -\rho\Psi \sin \Psi + \frac{(\rho\Psi \sin \Psi)^2}{2!} + \frac{\Psi^2}{2!} < -\frac{\pi\Psi}{2} \qquad (\xi > \xi_0). \tag{4.10}$$

Combining (4.9), (4.10), (2.15), (2.16) and (2.18) we obtain

$$G(\varphi)-G(\Psi) < -(\pi^2/3)(n/(\log n)^2) \qquad (n > n_0)$$

and (4.8) follows.

Our next lemma is immediate. From (1.6) we deduce

$$|H'''(\varphi)| < 2n \exp(\rho(\cos \varphi-\cos \Psi))(\log n)^2 \qquad (n > n_0).$$

Hence,

Lemma 4.1. *If*

$$|\varphi-\Psi| < 1/\log n, \tag{4.11}$$

then

$$|H'''(\varphi)| < 6n(\log n)^2. \tag{4.12}$$

5. Influence of the factor $h(z)$. Let $\lambda < +\infty$ be the order of $h(z)$. Hence, given $\eta > 0$, we have

$$|h'(z)| + |h(z)| < \exp(r^{\lambda+\eta}) \qquad (|z| = r > r_0(\eta)). \tag{5.1}$$

In the following lemma we summarize some well-known estimates derived from the elements of the theory of entire functions. For the sake of completeness we sketch a brief proof.

Lemma 5.1. *Let the zeros of* $h(z)$ *be denoted by* d_j, *and let them satisfy condition* I *of the introduction. Then, if* z *is confined to the strip*

$$|y-\pi| < \eta_1/2, \quad x > 0 \qquad (z = x + iy), \tag{5.2}$$

and if $\eta > 0$ *is given, we have, for* $|z| = r > r_0$,

$$|h(z)| > \exp(-r^{\lambda+\eta}), \tag{5.3}$$

$$|h'(z)/h(z)| < r^{\lambda+\eta}. \tag{5.4}$$

Proof. Start from the Poisson-Jensen-Nevanlinna identity

$$\log|h(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|h(Re^{i\theta})| \frac{R^2-r^2}{R^2 + r^2-2rR\cos(\theta-\varphi)}\, d\theta$$

$$-\sum_{|d_j|<R} \log\left|\frac{R^2-z\bar{d}_j}{R(z-d_j)}\right| \qquad (z = re^{i\varphi}, R = 2r),$$

where the quantities d_j are the zeros of $h(z)$. By assumption $|z-d_j| > \eta_1/2$, and hence, with the standard notations of Nevanlinna's theory

$$\log|h(z)| > -3m(2r, h) + n\left(2r, \frac{1}{h}\right)\log\left(\frac{\eta_1}{6r}\right) + O(1) \qquad (r \to \infty).$$

Inequality (5.3) is now immediate.

To prove (5.4) consider the familiar representation of the logarithmic derivative

$$\frac{h'(z)}{h(z)} = Q(z) + \frac{k_0}{z} + z^p \sum_{d_j \neq 0} \frac{1}{d_j^p(z-d_j)},$$

where k_0 and p ($\leq \lambda$) are integers and $Q(z)$ is a polynomial of degree $\leq \lambda - 1$.

Hence

$$\left|\frac{h'(z)}{h(z)}\right| \leq Kr^{\lambda-1} + \frac{2r^p}{\eta_1} \sum_{0<|d_j|<2r} \frac{1}{|d_j|^p} + 2r^p \sum_{|d_j|>2r} \frac{1}{|d_j|^{p+1}} + O(1)$$

$$\left(r \to \infty, |y-\pi| < \frac{1}{2}\eta_1\right), \tag{5.5}$$

and (5.4) follows. [Details of analogous estimates will be found in [9, p. 252].]

Lemma 5.2. *Let $q > \lambda$ (λ is the order of $h(z)$); q is otherwise arbitrary. Then*

I. *For* $0 \leq \varphi \leq \pi$, *we have*

$$\left|\frac{h(\tau + \rho e^{i\varphi})}{h(\tau + \rho e^{i\Psi})}\right| < \exp(3(\log n)^q) \qquad (n > n_0), \tag{5.6}$$

$$\left|\frac{W(\tau, n, \varphi)}{h(\tau + \rho e^{i\Psi})}\right| < \exp(3(\log n)^q) \qquad (n > n_0), \tag{5.7}$$

$$\left|\frac{Z(\tau, n, \varphi)}{h(\tau + \rho e^{i\Psi})}\right| < n\exp(3(\log n)^q) \qquad (n > n_0). \tag{5.8}$$

II. *If φ is further restricted by the conditions*

$$|\varphi-\Psi| \leq \frac{\eta_2}{3\pi}\Psi \qquad (\Psi = \Psi(\xi), \eta_2 = \min(1, \eta_1)), \tag{5.9}$$

then

$$\left|\frac{h(\tau + \rho e^{i\varphi})}{h(\tau + \rho e^{i\Psi})} - 1\right| \leq 2\rho^{q+1}|\varphi-\Psi|\exp(2\rho^{q+1}|\varphi-\Psi|) \qquad (n > n_0), \tag{5.10}$$

$$\left|\frac{W(\tau, n, \varphi)}{h(\tau + \rho e^{i\Psi})}\right| \leq A \exp(2\rho^{q+1}|\varphi-\Psi|)\left[|\varphi-\Psi| + \frac{\rho^q}{n}\right] \qquad (n > n_0), \tag{5.11}$$

$$\left|\frac{Z(\tau, n, \varphi)}{h(\tau + \rho e^{i\Psi})}\right| \leq A \exp(2\rho^{q+1}|\varphi-\Psi|)[n|\varphi-\Psi| + \rho^q] \qquad (n > n_0). \tag{5.12}$$

Proof. If $n > n_0$, the distance between a zero of $h(z)$ and $\tau + \rho e^{i\Psi}$ is at least $\eta_1/2$. Hence (5.3) is applicable and yields

$$|h(\tau + \rho e^{i\Psi})| \geq \exp(-(|\tau| + \rho)^q) \qquad (n > n_0). \tag{5.13}$$

In view of (5.1) and (2.18) we thus find

$$\left|\frac{h(\tau + \rho e^{i\varphi})}{h(\tau + \rho e^{i\Psi})}\right| + \left|\frac{h'(\tau + \rho e^{i\varphi})}{h(\tau + \rho e^{i\Psi})}\right| < \exp(3(\log n)^q) \qquad (n > n_0, 0 \leq \varphi \leq \pi). \tag{5.14}$$

The relation (5.6) is contained in (5.14). We now return to (3.5) and (3.6) and use (5.14), (2.21) and (2.22); this yields (5.7) and (5.8).

In order to take (5.9) into account we start from

$$\frac{h(\tau+\rho e^{i\varphi})}{h(\tau+\rho e^{i\Psi})}=\exp\left(i\int_{\Psi}^{\varphi}\frac{h'(\tau+\rho e^{i\theta})}{h(\tau+\rho e^{i\theta})}\rho e^{i\theta}\,d\theta\right), \tag{5.15}$$

and notice that in the above integral

$$|\operatorname{Im}\rho e^{i\theta}-\operatorname{Im}\rho e^{i\Psi}|=\rho|\sin\theta-\sin\Psi|<\rho\Psi\eta_2/3\pi,$$

$$|\operatorname{Im}\rho e^{i\theta}-\pi|<\eta_2/3+\varepsilon(n)<\eta_1/2 \qquad (n>n_0).$$

Hence our estimate (5.4) is valid and we thus find

$$\left|\frac{h'(\tau+\rho e^{i\theta})}{h(\tau+\rho e^{i\theta})}\right|<2\rho^q \qquad \left(n>n_0,\ |\theta-\Psi|<\frac{\eta_2}{3\pi}\Psi\right). \tag{5.16}$$

Combining (5.15), (5.16) and the elementary inequality $|e^w-1|<|w|e^{|w|}$, we obtain (5.10) as well as

$$\left|\frac{h(\tau+\rho e^{i\varphi})}{h(\tau+\rho e^{i\Psi})}\right|<\exp(2\rho^{q+1}|\varphi-\Psi|), \tag{5.17}$$

$$\left|\frac{h'(\tau+\rho e^{i\varphi})}{h(\tau+\rho e^{i\Psi})}\right|<2\rho^q\exp(2\rho^{q+1}|\varphi-\Psi|). \tag{5.18}$$

It is easily seen that, under the restrictions (5.9)

$$|e^t-e^\zeta|<Ae^{-\tau}n|\varphi-\Psi| \qquad (n>n_0),$$

$$|te^t-\zeta e^\zeta|<Ae^{-\tau}n(\log n)|\varphi-\Psi| \qquad (n>n_0).$$

Using these inequalities, (5.17), (5.18), (2.21) and (2.22), we readily deduce (5.11) and (5.12) from (3.5) and (3.6). This completes the proof of Lemma 5.2.

6. Decomposition of the interval of integration. Choose q ($>\lambda$) as in Lemma 5.2. Define

$$\theta_1=\Psi-\frac{(\log n)^{q/2}}{n^{1/2}}, \qquad \theta_2=\Psi+\frac{(\log n)^{q/2}}{n^{1/2}}, \tag{6.1}$$

$$X(\tau,n,\varphi)=\exp(H(\varphi)-H(\Psi)), \tag{6.2}$$

$$\Lambda(\tau,n)=\int_{\theta_1}^{\theta_2}\frac{h(\tau+\rho e^{i\varphi})}{h(\tau+\rho e^{i\Psi})}X(\tau,n,\varphi)\,d\varphi, \tag{6.3}$$

$$\Lambda_1(\tau,n)=\int_{\theta_1}^{\theta_2}\frac{W(\tau,n,\varphi)}{h(\tau+\rho e^{i\Psi})}X(\tau,n,\varphi)\,d\varphi, \tag{6.4}$$

$$\Lambda_2(\tau,n)=\int_{\theta_1}^{\theta_2}\frac{Z(\tau,n,\varphi)}{h(\tau+\rho e^{i\Psi})}X(\tau,n,\varphi)\,d\varphi. \tag{6.5}$$

With the same integrand as in (6.3) we write

$$\kappa_1(\tau,n)=\int_0^{\theta_1}, \qquad \kappa_2(\tau,n)=\int_{\theta_2}^{\pi}. \tag{6.6}$$

Similarly, with the same integrand as in (6.4),

$$\kappa_{11}(\tau, n) = \int_0^{\theta_1}, \qquad \kappa_{12}(\tau, n) = \int_{\theta_2}^{\pi}, \tag{6.7}$$

and with the same integrand as in (6.5),

$$\kappa_{21}(\tau, n) = \int_0^{\theta_1}, \qquad \kappa_{22}(\tau, n) = \int_{\theta_2}^{\pi}. \tag{6.8}$$

By assertion I of Lemma 5.2 we find

$$|\kappa_1| + |\kappa_2| + |\kappa_{11}| + |\kappa_{12}| + |\kappa_{21}| + |\kappa_{22}|$$
$$= \Delta < 3n \exp(3(\log n)^q)\left\{\int_0^{\theta_1} |X|\, d\varphi + \int_{\theta_2}^{\pi} |X|\, d\varphi\right\}. \tag{6.9}$$

Since, by (6.2), $|X| = \exp(G(\varphi) - G(\Psi))$, we deduce from inequalities (4.7) and (4.8),

$$\int_0^{\theta_1} |X|\, d\varphi + \int_{\theta_2}^{\pi} |X|\, d\varphi < \theta_1 \exp(G(\theta_1) - G(\Psi))$$
$$+ (2\Psi - \theta_2)\exp(G(\theta_2) - G(\Psi)) + \pi \exp\left(-\frac{3n}{(\log n)^2}\right). \tag{6.10}$$

To complete the estimates of this section we use Taylor's formula. By (1.11), (1.13) and Lemma 4.1 we find

$$H(\varphi) - H(\Psi) = -n(1 + \zeta)\frac{(\varphi - \Psi)^2}{2} + 2\omega(\varphi, \Psi) n (\log n)^2 (\varphi - \Psi)^3 \tag{6.11}$$

with $|\omega(\varphi, \Psi)| < 1$, provided $|\varphi - \Psi| < 1/\log n$.

In particular, with

$$|\theta_j - \Psi| = (\log n)^{q/2}/n^{1/2} \qquad (j = 1, 2),$$

we deduce from (6.11), (2.10) and (2.14),

$$G(\theta_j) - G(\Psi) < -\tfrac{1}{3}(\log n)^{1+q} \qquad (j = 1, 2;\ n > n_0).$$

Hence, by (6.9) and (6.10),

$$\Delta < 3\pi \exp\left(\log n + 3(\log n)^q - \tfrac{1}{3}(\log n)^{1+q}\right)$$
$$< \exp\left(-\tfrac{1}{4}(\log n)^{1+q}\right) \qquad (n > n_0). \tag{6.12}$$

7. The integrals Λ, Λ_1, Λ_2. We define

$$I = \int_{\theta_1}^{\theta_2} X(\tau, n, \varphi)\, d\varphi, \tag{7.1}$$

$$\tilde{I} = \int_{\theta_1}^{\theta_2} \left(\frac{h(\tau + \rho e^{i\varphi})}{h(\tau + \rho e^{i\Psi})} - 1\right) X\, d\varphi, \tag{7.2}$$

so that, by the definition of $\Lambda(\tau, n)$, in (6.3),

$$\Lambda(\tau, n) = I + \tilde{I}. \tag{7.3}$$

Since

$$|\varphi-\Psi| < (\log n)^{q/2}/n^{1/2} \qquad (\theta_1 < \varphi < \theta_2) \tag{7.4}$$

we note that (5.10) yields

$$|\tilde{I}| < 3(\log n)^{3q/2+1} n^{-1/2} \int_{\theta_1}^{\theta_2} |X(\tau, n, \varphi)|\, d\varphi \qquad (n > n_0). \tag{7.5}$$

From (7.1) and (4.7)

$$|I| < \int_{\theta_1}^{\theta_2} |X(\tau, n, \varphi)|\, d\varphi < \int_{\theta_1}^{\theta_2} \exp(G(\varphi)-G(\Psi))\, d\varphi < (\theta_2-\theta_1), \tag{7.6}$$

which, in view of (7.4) and (7.5), implies

$$|\tilde{I}| < 6(\log n)^{2q+1} n^{-1} \qquad (n > n_0). \tag{7.7}$$

To obtain a good approximation of I, put

$$\Omega = -n \log n \left(\frac{1+\zeta}{\log n} \right) \frac{(\varphi-\Psi)^2}{2}. \tag{7.8}$$

An inspection of our approximation for ζ shows that we could, if necessary, evaluate

$$\frac{1+\zeta}{\log n} = 1 + \mathcal{E}_n(\tau) \tag{7.9}$$

with considerable precision; for our purpose it suffices to note that

$$|\mathcal{E}_n(\tau)| = \varepsilon(n) \qquad (n \to \infty). \tag{7.10}$$

Using (6.11), (7.8), (7.9) and the definition of I, we see that

$$I = I(\tau, n) = \int_{\theta_1}^{\theta_2} e^{\Omega} d\varphi + \int_{\theta_1}^{\theta_2} e^{\Omega}\left(-1 + \exp\left[2\omega n(\log n)^2(\varphi-\Psi)^3\right]\right) d\varphi. \tag{7.11}$$

The modulus of the latter integral cannot exceed

$$5(\log n)^{2q+2}/n. \tag{7.12}$$

To obtain the value of the first integral in (7.11) we introduce, as a new variable of integration,

$$\sigma = \left(\frac{n \log n}{2} \right)^{1/2} (\varphi-\Psi)$$

and find

$$\int_{\theta_1}^{\theta_2} e^{\Omega}\, d\varphi = \frac{2^{3/2}}{(n \log n)^{1/2}} \int_0^{\beta(n)} e^{-\sigma^2(1+\mathcal{E}_n(\tau))}\, d\sigma, \tag{7.13}$$

with

$$\beta(n) = 2^{-1/2}(\log n)^{(q+1)/2}.$$

Now

$$\left| \int_{\beta(n)}^{\infty} e^{-\sigma^2(1+\mathcal{E}_n(\tau))}\, d\sigma \right| < \int_{\beta(n)}^{\infty} e^{-(1/2)\sigma^2} \sigma d\sigma = \exp\left(-\tfrac{1}{4}(\log n)^{q+1}\right) \tag{7.14}$$

and by an elementary application of Cauchy's theorem,

$$\int_0^{\infty} e^{-\sigma^2(1+\mathcal{E}_n(\tau))}\, d\sigma = \frac{1}{2}\left(\frac{\pi}{1+\mathcal{E}_n(\tau)} \right)^{1/2}. \tag{7.15}$$

Combining (7.13), (7.14) and (7.15) we are led to

$$\int_{\theta_1}^{\theta_2} e^{\Omega}\,d\varphi = \left\{\frac{2\pi}{(n\log n)(1+\mathfrak{S}_n(\tau))}\right\}^{1/2} + \omega\exp\left(-\frac{1}{4}(\log n)^{q+1}\right) \qquad (|\omega|<1,\ n>n_0),$$

and returning to (7.11) we find

$$I(\tau,n) = (2\pi)^{1/2}\{n(\log n)(1+\mathfrak{S}_n(\tau))\}^{-1/2} + 6\omega\frac{(\log n)^{2q+2}}{n} \qquad (|\omega|<1,\ n>n_0). \tag{7.16}$$

From (6.3), (6.6) and (7.3)

$$J(\tau,n) = h(\tau+\rho e^{i\Psi})\{I(\tau,n)+\tilde{I}+\kappa_1+\kappa_2\}, \tag{7.17}$$

and hence

$$J(\tau,n) = h(\tau+\rho e^{i\Psi})\{(2\pi)^{1/2}\{n(\log n)(1+\mathfrak{S}_n(\tau))\}\}^{-1/2} \cdot\left(1+\frac{A\omega(\log n)^{2q+(5/2)}}{n^{1/2}}\right) \qquad (|\omega|<1,\ n>n_0). \tag{7.18}$$

[An inspection of (7.16), (7.7) and (6.12) will disclose the validity of the error term in (7.18).]

Crude estimates of $\partial J/\partial n$, $\partial J/\partial\tau$ are sufficient for our purpose. From (6.5) and (5.12) we deduce

$$|\Lambda_2(\tau,n)| < An^{1/2}(\log n)^{q/2}\int_{\theta_1}^{\theta_2}|X|\,d\varphi < A(\log n)^q$$

and in view of (6.8), (6.12) and (3.4),

$$|\partial J/\partial\tau| < A|h(\tau+\rho e^{i\Psi})|(\log n)^q. \tag{7.19}$$

Similarly,

$$|\partial J/\partial n| < A|h(\tau+\rho e^{i\Psi})|(\log n)^q n^{-1}. \tag{7.20}$$

By (7.18), (7.19) and (7.20)

$$\left|\frac{1}{J}\,\frac{\partial J}{\partial\tau}\right| < An^{1/2}(\log n)^{q+(1/2)} \qquad (n>n_0), \tag{7.21}$$

$$\left|\frac{1}{J}\,\frac{\partial J}{\partial n}\right| < An^{-1/2}(\log n)^{q+(1/2)} \qquad (n>n_0). \tag{7.22}$$

8. Zeros and sign of $f^{(n)}(\tau)$. Rewrite (1.14) as

$$f^{(n)}(\tau) = n!\pi^{-1}\rho^{-n}\mathrm{Re}\{\exp(H(\Psi)+\log J(\tau,n))\}$$
$$= n!\pi^{-1}\rho^{-n}|J(\tau,n)|e^{G(\Psi)}\cos(U(\tau,n)), \tag{8.1}$$

where

$$U(\tau,n) = L(\Psi) + \mathrm{Im}\{\log J(\tau,n)\}. \tag{8.2}$$

The possibility of using (8.2) as ξ varies from $ne^{-\tau_1}$ to $ne^{-\tau_2}$ (or from $n_1e^{-\tau}$ to $n_2e^{-\tau}$) depends on the remark that for all such values of ξ,

$$J(\tau, n) \neq 0. \tag{8.3}$$

This may be seen by returning to (7.18) and studying the behavior of the factor

$$h(\tau + \rho e^{i\Psi}) = h(\tau + \zeta(\xi)). \tag{8.4}$$

In (8.4),

$$\tau + \zeta(\xi) = \tau + x(\xi) + iy(\xi) \qquad (\xi = ne^{-\tau}),$$

where, by (2.15),

$$\lim_{n\to\infty} \operatorname{Im}\{\tau + \zeta(\xi)\} = \pi, \tag{8.5}$$

and by (2.14) and (1.16),

$$\tau + x(\xi) = \log n - \log\log n + \varepsilon(n). \tag{8.6}$$

Hence $h(\tau + \zeta(\xi)) \neq 0$, by our assumption I concerning the zeros of $h(z)$, and (8.3) follows.

We have thus proved that:

The real zeros and the sign of $f^{(n)}(\tau)$ *coincide with those of* $\cos(U(\tau, n))$.

An explicit expression for $L(\Psi)$ follows from (4.6) and (2.10):

$$L(\Psi) = n\left\{\operatorname{Im}\left(\zeta + \frac{1}{\zeta}\right) - \pi\right\} = n\left\{\left(y(\xi) - \frac{y(\xi)}{\rho^2(\xi)}\right) - \pi\right\}. \tag{8.7}$$

Using the approximation (2.15) and (1.16) we find

$$L(\Psi(\xi)) = -(n\pi/\log n)(1 + \varepsilon(n)). \tag{8.8}$$

Taking into account (2.18), (2.20) and (2.21), we obtain

$$\frac{d}{d\xi}\left(y(\xi) - \frac{y(\xi)}{\rho^2(\xi)} - \pi\right) = \dot{y}(\xi) - \frac{\dot{y}(\xi)}{\rho^2} + 2\frac{\dot{\rho}}{\rho}\,\frac{y(\xi)}{\rho^2}$$

$$= \frac{\pi}{\xi(x(\xi))^2} + \omega\frac{A}{\xi(\log\xi)^3} \qquad (|\omega| < 1, n > n_0),$$

and hence, in view of (8.7) and (2.15),

$$\frac{\partial L(\Psi(\xi))}{\partial n} = \left(y - \pi - \frac{y}{\rho^2}\right) + \frac{\pi}{x^2} + \omega\frac{A}{(\log\xi)^3}$$

$$= -\frac{\pi}{1 + \log\xi} + \frac{\omega A(\log\log\xi)^2}{(\log\xi)^2} \qquad (|\omega| < 1, n > n_0). \tag{8.9}$$

By (8.2), (7.22), and (8.9) we see that

$$\partial U/\partial n = -(\pi/\log n)(1 + \varepsilon(n)) < 0 \qquad (n > n_0). \tag{8.10}$$

The arguments which lead to (8.9) also show that

$$\frac{\partial L(\Psi(\xi))}{\partial\tau} = -n\left(\frac{\pi}{x^2} + \omega\frac{A}{(\log\xi)^3}\right) \qquad (|\omega| < 1, n > n_0) \tag{8.11}$$

and by (7.21)

$$\partial U/\partial\tau = -\left(n\pi/(\log n)^2\right)(1 + \varepsilon(n)) < 0 \qquad (n > n_0). \tag{8.12}$$

9. Variation of U and L: Proof of Theorem 1. Let $n > n_0$ be fixed and let τ increase from τ_1 to τ_2. Since n is fixed we simplify our notation and write $U(\tau)$ instead of $U(\tau, n)$. An inspection of (8.12) shows that $U(\tau)$ is strictly decreasing from $U(\tau_1)$ to $U(\tau_2)$. Hence, returning to (8.1), we see that all the zeros of $f^{(n)}(\tau)$ in the interval $[\tau_1, \tau_2]$ *are necessarily simple* and that the inequalities

$$-1 \leq \frac{U(\tau_1)-U(\tau_2)}{\pi} - N(\tau_1, \tau_2; n) < 1 \qquad (n > n_0), \tag{9.1}$$

must be satisfied.

Write

$$\xi_1 = ne^{-\tau_1}, \qquad \xi_2 = ne^{-\tau_2}, \tag{9.2}$$

$$T(\tau_1, \tau_2; n) = (L(\Psi(\xi_1)) - L(\Psi(\xi_2)))/\pi, \tag{9.3}$$

$$E(\tau_1, \tau_2; n) = (1/\pi)|\mathrm{Im} \log J(\tau_1, n) - \mathrm{Im} \log J(\tau_2, n)|. \tag{9.4}$$

The evaluation of the right-hand side of (9.4) requires some caution. As may be seen from (8.2), the correct value of $E(\tau_1, \tau_2; n)$ is to be obtained by letting τ vary continuously from τ_1 to τ_2. We shall use the estimate

$$E(\tau_1, \tau_2; n) \leq \frac{1}{\pi}\int_{\tau_1}^{\tau_2}\left|\frac{1}{J}\,\frac{\partial J}{\partial \tau}\right| d\tau$$

$$\leq (\tau_2-\tau_1)An^{1/2}(\log n)^{q+(1/2)} \qquad (n > n_0) \tag{9.5}$$

(derived from (7.21)), thus avoiding possible misinterpretations of the ambiguous symbols

$$\mathrm{Im}(\log(J(\tau_1, n)/J(\tau_2, n))).$$

To complete our proof of Theorem 1, we still require the evaluation of $T(\tau_1, \tau_2; n)$. By (8.11) and the mean-value theorem,

$$T(\tau_1, \tau_2; n) = n(\tau_2-\tau_1)\left(\frac{1}{x^2} + \omega\frac{A}{(\log \xi)^3}\right) \tag{9.6}$$

where $x = \log n - \log\log n - \tau + \varepsilon(n)$ and τ is a suitable value such that $-D \leq \tau_1 < \tau < \tau_2 \leq D$. Hence,

$$T(\tau_1, \tau_2; n) = \frac{n(\tau_2-\tau_1)}{(\log n - \log_2 n)^2} + \omega A(D+1)\frac{n}{(\log n)^3} \qquad (n > n_0). \tag{9.7}$$

Since

$$\frac{U(\tau_1)-U(\tau_2)}{\pi} = T(\tau_1, \tau_2; n) + \omega E(\tau_1, \tau_2; n), \tag{9.8}$$

inequality (7) of Theorem 1 now follows from this relation, (9.1), (9.5) and (9.7).

10. Variation of U and L for τ fixed. The real quantity τ is fixed; we simplify our notation by writing $U(n)$ instead of $U(ne^{-\tau})$ and setting

$$\chi_1 = n_1e^{-\tau}, \qquad \chi_2 = n_2e^{-\tau}. \tag{10.1}$$

Although (8.1) is valid only if n is a positive integer, the consideration of $U(\tau, n)$ in (8.2), as well as the study of $\cos(U(\tau, n))$, may be performed for a continuous variation of $n > 0$.

The relation analogous to (9.3), (9.4) and (9.8) will now be

$$\tilde{T}(\tau; n_1, n_2) = (L(\Psi(\chi_1)) - L(\Psi(\chi_2)))/\pi, \tag{10.2}$$

$$\tilde{E}(\tau; n_1, n_2) = (1/\pi)|\mathrm{Im}\log J(\tau, n_1) - \mathrm{Im}\log J(\tau, n_2)|, \tag{10.3}$$

$$(U(n_1) - U(n_2))/\pi = \tilde{T}(\tau; n_1, n_2) + \omega\tilde{E}(\tau; n_1, n_2). \tag{10.4}$$

The estimate

$$\tilde{E}(\tau; n_1, n_2) \leqslant \frac{1}{\pi}\int_{n_1}^{n_2}\left|\frac{1}{J}\,\frac{\partial J}{\partial n}\right| dn$$

$$\leqslant (n_2 - n_1)An_1^{-1/2}(\log n_1)^{q+(1/2)} \qquad (n_2 > n_1 > n_0) \tag{10.5}$$

corresponds to (9.5) and is obtained in the same way (using (7.22) instead of (7.21)).

Beside (10.5) we also have

$$\tilde{E}(\tau, n_1, n_2) \leqslant An_2^{1/2}(\log n_2)^{q+(1/2)} \qquad (n_2 > n_1 > n_0). \tag{10.6}$$

Now (8.2), (8.8) and (10.6) yield

$$U(n_1) - U(n_2) = L(\Psi(\chi_1)) + \frac{n_2\pi}{\log n_2}(1 + \varepsilon(n_2)) + \omega An_2^{1/2}(\log n_2)^{q+(1/2)}$$

$$(n_2 \to \infty). \tag{10.7}$$

From (8.10) we deduce that, for τ fixed and $n > n_0$, $U(n)$ is a continuous, strictly decreasing unbounded function of n. It will assume exactly once every large negative value. Hence it will be possible to define a strictly increasing, unbounded sequence

$$\nu_0, \nu_1, \nu_2, \nu_3, \ldots \tag{10.8}$$

such that

$$U(\nu_k) = -\pi\left(l + k + \tfrac{1}{2}\right) \qquad (k = 0, 1, 2, 3, \ldots). \tag{10.9}$$

Take l, in (10.9) to be a large, positive, even integer; this ensures that

$$\mathrm{sign}\{\cos(U(\nu))\} = (-1)^{k+1} \tag{10.10}$$

for

$$\nu_k < \nu < \nu_{k+1}. \tag{10.11}$$

We may use (10.7) with $n_1 = \nu_0$, $n_2 = \nu_k$. Then by (10.9),

$$\pi k = L(\Psi(e^{-\tau}\nu_0)) + \frac{\pi\nu_k}{\log \nu_k}(1 + \varepsilon(\nu_k)) + \omega A\nu_k^{1/2}(\log \nu_k)^{q+(1/2)} \qquad (k \to \infty),$$

and hence

$$\lim_{k\to\infty} \frac{k \log \nu_k}{\nu_k} = 1. \tag{10.12}$$

With minor modifications the method gives us the asymptotic behavior of the differences $\nu_{k+1} - \nu_k$. We first note that the analogue of (9.6) (which follows from (8.9) and from the mean-value theorem) is

$$\tilde{T}(\tau; n_1, n_2) = (n_2 - n_1)\left\{\frac{1}{1+\log\xi} + \omega\frac{A(\log\log\xi)^2}{(\log\xi)^2}\right\}, \tag{10.13}$$

where

$$\xi = \log n - \tau, \tag{10.14}$$

and n is a suitable value such that

$$n_0 < n_1 < n < n_2. \tag{10.15}$$

Select in (10.13),

$$n_1 = \nu_k, \qquad n_2 = \nu_k + \mu_k \log \nu_k \tag{10.16}$$

where $\{\mu_k\}$ is an arbitrary positive sequence.

I. If $\mu_k < \frac{3}{2}$, (10.13) yields

$$\tilde{T}(\tau; \nu_k, \nu_k + \mu_k \log \nu_k) = \mu_k(1 + \varepsilon(\nu_k)) \qquad (k \to \infty), \tag{10.17}$$

and hence, in view of (10.4) and (10.5),

$$\frac{U(\nu_k) - U(\nu_k + \mu_k \log \nu_k)}{\pi} = \mu_k(1 + \varepsilon(\nu_k)) + \varepsilon(\nu_k) \qquad (k \to \infty). \tag{10.18}$$

II. If $\mu_k > \frac{3}{2}$, then

$$\frac{U(\nu_k) - U(\nu_k + \mu_k \log \nu_k)}{\pi} > \frac{U(\nu_k) - U\left(\nu_k + \frac{3}{2}\log \nu_k\right)}{\pi}$$

$$= \tfrac{3}{2}(1 + \varepsilon(\nu_k)) \qquad (k \to \infty). \tag{10.19}$$

[The inequality in (10.19) follows from the fact that $U(n)$ is a strictly decreasing function of n.]

We may now define the sequence μ_k by the relations

$$\nu_{k+1} = \nu_k + \mu_k \log \nu_k. \tag{10.20}$$

In view of (10.9) the relations (10.18) and (10.19) become, respectively,

$$1 = \mu_k(1 + \varepsilon(\nu_k)) + \varepsilon(\nu_k) \qquad \left(0 < \mu_k < \tfrac{3}{2}\right) \tag{10.21}$$

and

$$1 > \tfrac{3}{2}(1 + \varepsilon(\nu_k)) \qquad \left(\tfrac{3}{2} < \mu_k\right).$$

The latter relation is clearly impossible for $k > k_0$ and we must therefore have, in (10.20),

$$\mu_k < \tfrac{3}{2} \qquad (k > k_0).$$

Hence (10.21) must be valid for large values of k and consequently (10.20) yields

$$\frac{\nu_{k+1} - \nu_k}{\log \nu_k} \to 1 \qquad (k \to +\infty). \tag{10.22}$$

11. Proof of Theorem 2. The construction of our sequence (10.8) and the relations (10.12) and (10.22) prove assertions I and II of Theorem 2. The remaining assertions III and IV are obvious consequences of (10.10) and (10.11).

REFERENCES

1. A. Edrei and G. R. MacLane, *On the zeros of the derivatives of an entire function*, Proc. Amer. Math. Soc. **8** (1957), 702–706.

2. S. Hellerstein and J. Williamson, *Derivatives of entire functions and a question of Pólya*, Trans. Amer. Math. Soc. **227** (1977), 227–249.

3. ______, *Derivatives of entire functions and a question of Pólya*. II, Trans. Amer. Math. Soc. **234** (1977), 497–503.

4. L. Moser and M. Wyman, *An asymptotic formula for the Bell numbers*, Trans. Roy. Soc. Canada **49** (1955), 49–54.

5. G. Pólya, *On the zeros of the derivatives of a function and its analytic character*, Bull. Amer. Math. Soc. **49** (1943), 178–191.

6. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. II, Springer-Verlag, Berlin and New York, 1925.

7. C. Prather, *On the zeros of successive derivatives of some entire functions* (unpublished).

8. G. Szegö, *Orthogonal polynomials*, 3rd ed., Amer. Math. Soc. Colloq. Publ., no. 23, Amer. Math. Soc., Providence, R. I., 1967.

9. E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Univ. Press, London, 1952.

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13210